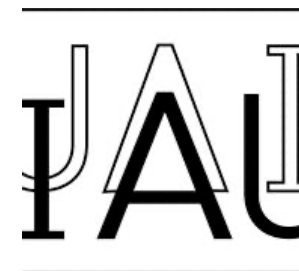
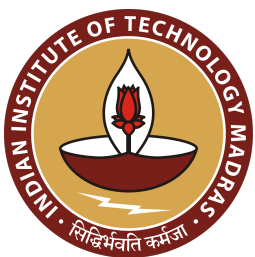


Elements of PN-MPM formalism

Chandra Kant Mishra
IIT Madras, Chennai (India)

Foundations of Gravitational Waves: Theory, Sources, and Detection
IUCAA, May 19-June 4, 2026

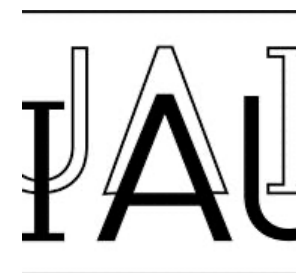
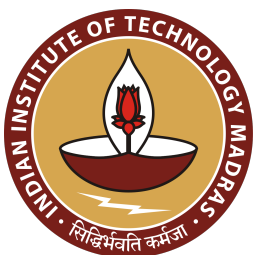


GW sources, modeling and detection : Post-Newtonian Theory



Chandra Kant Mishra
IIT Madras, Chennai (India)

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IUCAA, May 19-June 4, 2026



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- ◆ *Post-Newtonian Theory for Gravitational Waves*, Luc Blanchet, [arxiv:1310.1528](#) — **Blanchet Living Review**
- ◆ *Gravitational waves from inspiralling compact binaries: Energy loss and waveform to second--post-Newtonian order*, Luc Blanchet, Thibault Damour, Bala R Iyer, [gr-qc/9501029](#) — **BDI**
- ◆ *A Comparison of search templates for gravitational waves from binary inspiral*, Thibault Damour, Bala R. Iyer, B S Sathyaprakash, [gr-qc/0010009](#) — **DIS**

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- ◆ *Post-Newtonian Theory for Gravitational Waves*, Luc Blanchet, [arxiv:1310.1528](https://arxiv.org/abs/1310.1528) — **Blanchet Living Review**
- ◆ *Multipole expansions of gravitational radiation*, Kip Thorne, [Rev. Mod. Phys. 52, 299](https://www.jstor.org/stable/2372227) — **Thorne (1980)**

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- ◆ *Gravitational waves from inspiralling compact binaries: Energy loss and waveform to second--post-Newtonian order*, Luc Blanchet, Thibault Damour, Bala R Iyer, [gr-qc/9501029](#) — **BDI**
- ◆ *Gravitational-Radiation Damping of Compact Binary Systems to Second Post-Newtonian order*, Luc Blanchet, Thibault Damour, Bala R Iyer, Clifford M. Will, Alan G. Wiseman, <https://arxiv.org/abs/gr-qc/9501027> — **BDI-WW**

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- ◆ *A Comparison of search templates for gravitational waves from binary inspiral*, Thibault Damour, Bala R. Iyer, B S Sathyaprakash, [gr-qc/0010009](#) — **DIS**
- ◆ *Gravitational-Wave Inspiral of Compact Binary Systems to 7/2 Post-Newtonian Order*, Luc Blanchet, Guillaume Faye, Bala R. Iyer, Benoit Joguet, [gr-qc/0105099](#) — **BFIJ**

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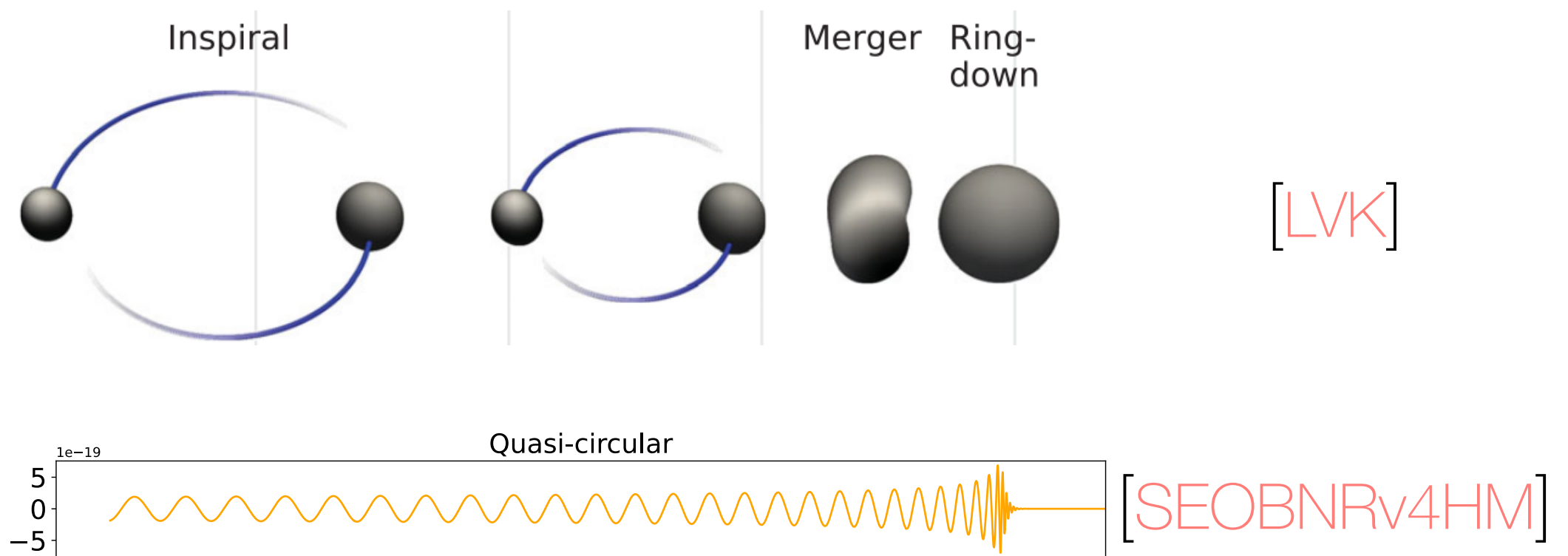
- ◆ Chapter 5, *Gravitational Waves, Volume 1 : Theory and Experiment*, Michele Maggiore
- ◆ *Gravity : Newtonian, Post-Newtonian, Relativistic*, Eric Poisson & Clifford M. Will

Outline - Lecture 1

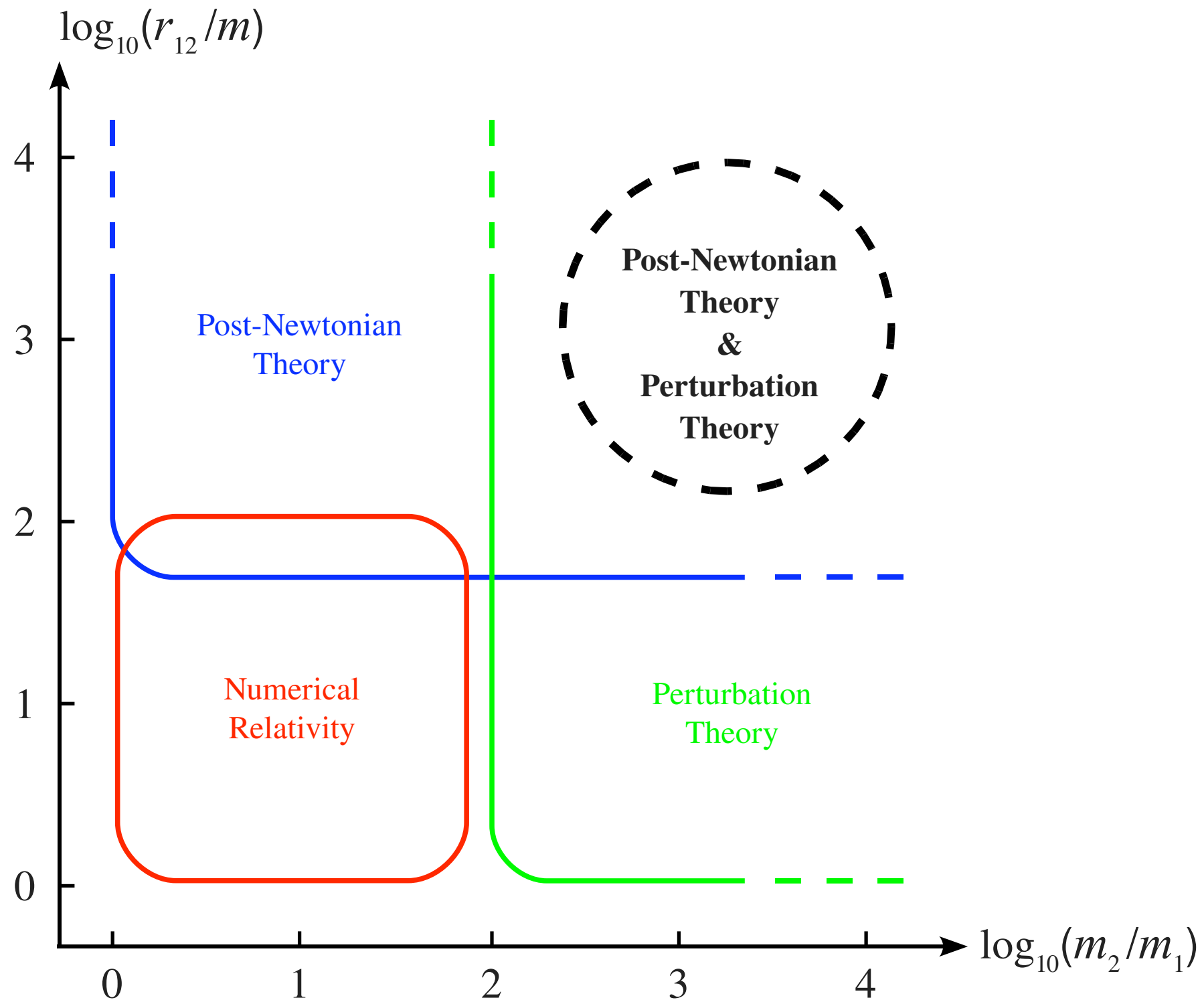
- ◆ The Post-Newtonian (PN) approximation — background, motivation
- ◆ The Post-Newtonian Multi polar-Post-Minkowskian (PN-MPM) approach for waveform generation

The post-Newtonian
approximation — background,
motivation and current status

BBH Waveforms



Modelling schemes



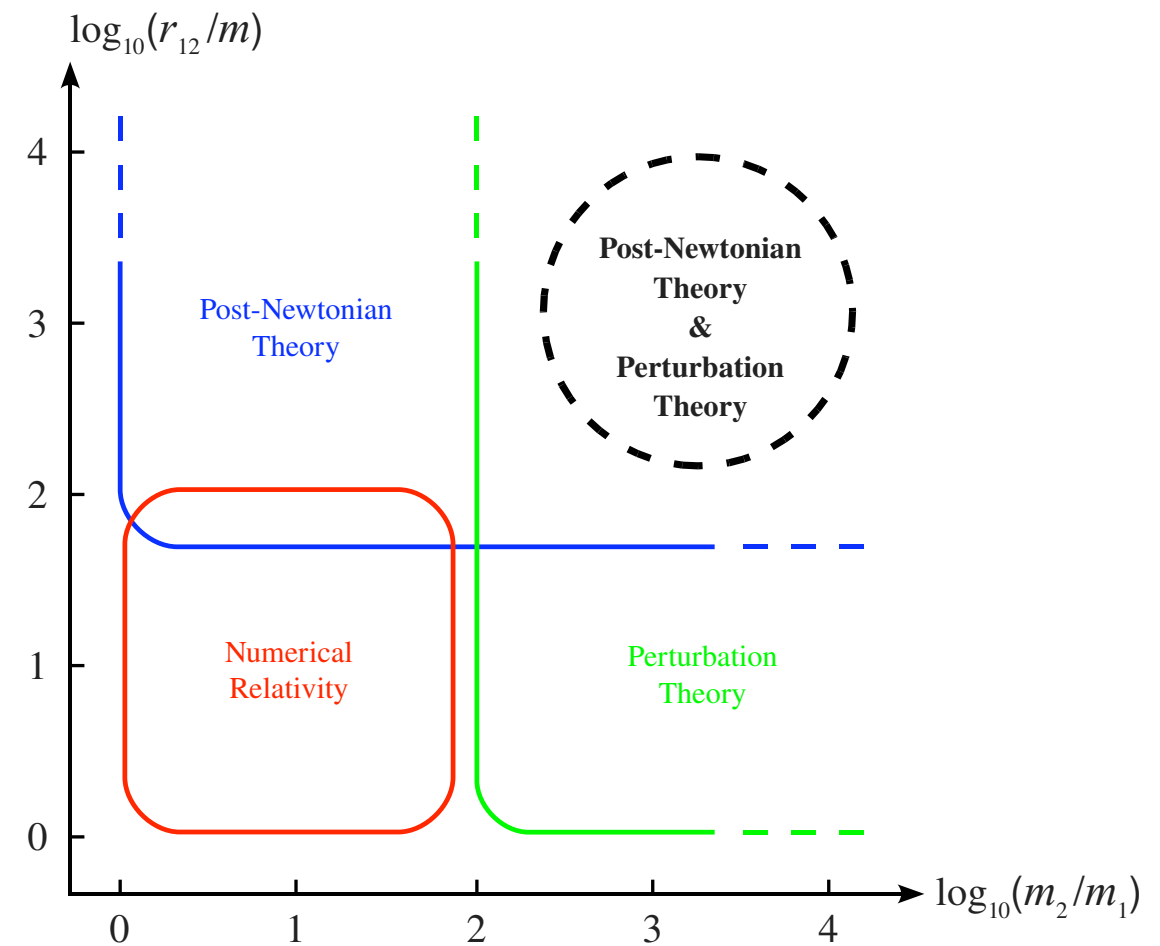
[Image Credit: A. Le Tiec]

The post-Newtonian Approach

$$r_{12} \gg M$$

$$r_{12} \gg R_A$$

- ◆ Weakly bound
- ◆ Composed of point masses



The post-Newtonian Approach

$$r_{12} \gg GM/c^2 \quad \& \quad \frac{GM}{c^2 r_{12}} \sim \frac{v^2}{c^2} \ll 1$$

Example : SCH metric

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + d\Sigma^2 \\ &\simeq - \left(1 - \frac{v_{\text{esc}}^2}{c^2} \right) c^2 dt^2 + \left(1 + \frac{v_{\text{esc}}^2}{c^2} \right) dr^2 + d\Sigma^2 \end{aligned}$$

The post-Newtonian Approach

Example : Precession of Mercury's orbit — Weakly bound, circular orbit

$$\Delta\phi \sim 2\pi \left(1 + \frac{3GM}{c^2 r} \right) \sim 2\pi \left(1 + \frac{3v^2}{c^2} \right) \quad [\text{change per orbit}]$$

$$v = r\omega \implies v^2/c^2 = \left(\frac{GM\omega}{c^3} \right)^{2/3} \\ \sim 10^{-8} \quad [\text{Mercury's orbit}]$$

- ◆ Clearly a small correction to the leading effect justifying the approximation used.

The post-Newtonian Approach

- ◆ Post-Newtonian corrections, simply put, are Taylor expansions in “small” parameters such as v^2/c^2 and thus are valid for “weak-fields” and “small-velocities”.
- ◆ Corrections of the order, $(v/c)^{2n}$, are referred to as being n -PN accurate.

$$\frac{GM}{c^2 r_{12}} \sim \frac{v^2}{c^2} \sim \text{1PN} , \quad \left(\frac{GM}{c^2 r_{12}} \right)^2 \sim \frac{v^4}{c^4} \sim \text{2PN} \quad \text{etc.}$$

The post-Newtonian Approach

- ◆ Higher order corrections help tackle “mildly” relativistic fields but as can be expected the approximation would break somewhere.
- ◆ Also, results computed at the highest order need not have the highest accuracy.
- ◆ PN series expansions converge slowly and become challenging at high orders and thus not ideal to deal with “large” mass ratios ($m_2 \gg m_1$).

The post-Newtonian Approach

- ◆ Nevertheless, are useful for accurate modelling of “low-mass” compact binaries composed of neutron stars and/or black holes.
- ◆ Help validating other perturbative approaches such as self-force approach.
- ◆ Used as inputs for development of complete inspiral-merger-ringdown models such as Effective-One-Body (EOB) and Phenomenological Models (Phenom).
- ◆ And in performing (parametrized) Tests of General Relativity.

The PN-MPM approach

PN-MPM approach — A soup of approximations

Near Zone

- ◆ “weak-fields” and “slow motion” — final orbits can be at half the speed of light — Post-Newtonian sources.

$$\frac{GM}{c^2 r_{12}} \sim \mathcal{O} \left(\frac{v^2}{c^2} \right)$$

- ◆ Binary components are treated as Point-Particles

$$R_A \ll \lambda$$

PN-MPM approach — A soup of approximations

Far-Zone/Exterior field

- ◆ Far-zone expansion

$$h \sim \mathcal{O}\left(\frac{1}{R}\right)$$

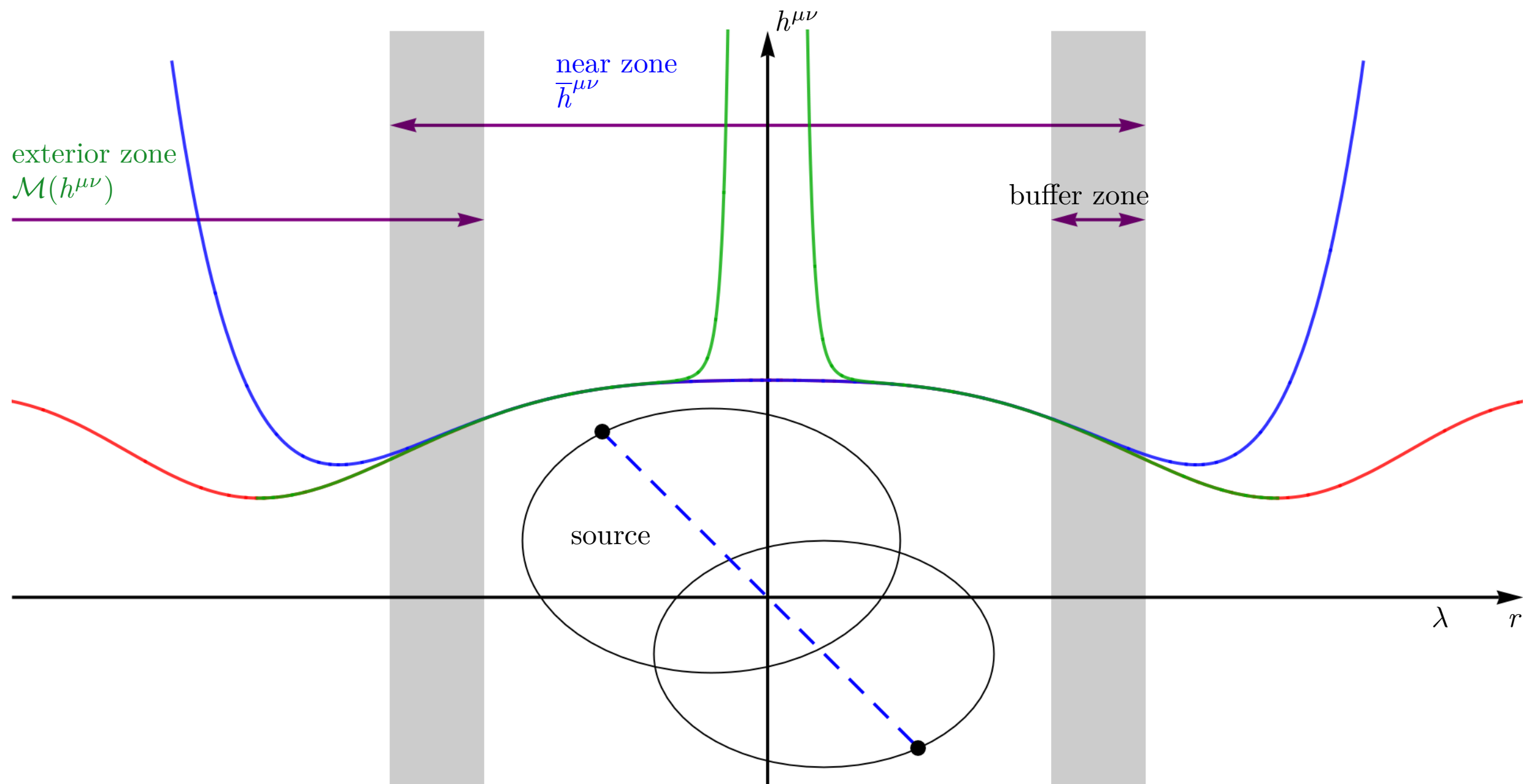
- ◆ Multipole expansion

$$I^\ell \sim \mathcal{O}(r_{12}^\ell)$$

- ◆ Post-Minkowskian expansion

$$h \sim \mathcal{O}(G^n)$$

PN-MPM Solution



Outline - Lecture 2

- ◆ The **PN-MPM** approach — the formalism
 - ◆ Quadrupole formula — brief review
 - ◆ General structure of radiated fields and fluxes
 - ◆ Connecting Radiative and source multipole moments — the PN-MPM solution

The **PN-MPM** approach — the formalism

The Quadrupole formula

GW Radiation field

$$h_{ij}^{\text{TT}} = -\frac{2G}{c^4 R} \perp_{ijab}(\mathbf{N}) \left\{ \frac{d^2 Q_{ab}}{dT^2}(T - R/c) + \mathcal{O}\left(\frac{1}{c}\right) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Quadrupole moment tensor

$$Q_{ij}(t) = \int_{\text{source}} d^3\mathbf{x} \rho(\mathbf{x}, t) \left(x_i x_j - \frac{1}{3} \delta_{ij} \mathbf{x}^2 \right)$$

Note: $T - R/c$: Retarded time in **far-zone** coordinates $(\mathbf{X}, T)$; also referred to as **radiative coordinates**. $\mathbf{N}$ is a unit vector along the line joining the source and the observer.

The Quadrupole formula

GW Radiation field

$$h_{ij}^{\text{TT}} = -\frac{2G}{c^4 R} \perp_{ijab}(\mathbf{N}) \left\{ \frac{d^2 Q_{ab}}{dT^2}(T - R/c) + \mathcal{O}\left(\frac{1}{c}\right) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

GW energy flux

$$\mathcal{F} \equiv \left(\frac{dE}{dT} \right)^{\text{GW}} = \frac{G}{c^5} \left\{ \frac{1}{5} \frac{d^3 Q_{ab}}{dT^3} \frac{d^3 Q_{ab}}{dT^3} + \mathcal{O}\left(\frac{1}{c^2}\right) \right\}$$

Note: $T - R/c$: Retarded time in **far-zone** coordinates $(\mathbf{X}, T)$; also referred to as **radiative coordinates**. $\mathbf{N}$ is a unit vector along the line joining the source and the observer.

General structure of Radiation Field

Leading GW Radiation field

$$h_{ij}^{\text{TT}} = -\frac{2G}{c^4 R} \perp_{ijab}(\mathbf{N}) \left\{ \frac{d^2 Q_{ab}}{dT^2} (T - R/c) + \mathcal{O}\left(\frac{1}{c}\right) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

GW Radiation Field

$$h_{ij}^{\text{TT}}(U, \mathbf{X}) = -\frac{4G}{c^2 R} \perp_{ijab}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{abL-2}(U) - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} V_{b)dL-2}(U) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Note:

$$U \equiv T - R/c \quad \& \quad L = i_1 \cdots i_\ell \quad \& \quad U_{ij} = Q_{ij}^{(2)} + \mathcal{O}(1/c^2)$$

General structure of Energy Flux

Leading GW energy flux

$$\mathcal{F} \equiv \left(\frac{dE}{dT} \right)^{\text{GW}} = \frac{G}{c^5} \left\{ \frac{1}{5} \frac{d^3 Q_{ab}}{dT^3} \frac{d^3 Q_{ab}}{dT^3} + \mathcal{O} \left(\frac{1}{c^2} \right) \right\}$$

GW Energy Flux

$$\mathcal{F} \equiv \left(\frac{dE}{dU} \right)^{\text{GW}} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ a_{\ell} U_L^{(1)} U_L^{(1)} + \frac{b_{\ell}}{c^2} V_L^{(1)} V_L^{(1)} \right\},$$

Note:

$$U_L = U_{i_1 \dots i_{\ell}} \quad \& \quad V_L = V_{i_1 \dots i_{\ell}} \quad \& \quad U_{ij}^{(1)} = Q_{ij}^{(3)} + \mathcal{O}(1/c^2)$$

Connecting Radiative and Source multipoles

$$U_{ij} = M_{ij}^{(2)}(U) + \sum_{n=3}^9 U_{ij}^{\frac{n}{2}\text{PN}}(U) + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

$$M_{ij} = I_{ij} + \frac{4G}{c^5} \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right] + \mathcal{O}\left(\frac{v^7}{c^7}\right)$$

Note:

$$M_L = M_{i_1 \dots i_\ell} \quad \& \quad M_{ij} = I_{ij} + \mathcal{O}(1/c^5) \quad \& \quad M_{ij}^{(2)} = I_{ij}^{(2)} + \mathcal{O}(1/c^5)$$

Connecting Radiative and Source multipoles

$$U_{ij} = M_{ij}^{(2)}(U) + \sum_{n=3}^9 U_{ij}^{\frac{n}{2}\text{PN}}(U) + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

$$U_{ij} = Q_{ij}^{(2)} + \mathcal{O}(1/c^2)$$

Note:

$$M_L = M_{i_1 \dots i_\ell} \quad \& \quad M_{ij} = I_{ij} + \mathcal{O}(1/c^5) \quad \& \quad M_{ij}^{(2)} = I_{ij}^{(2)} + \mathcal{O}(1/c^5)$$

Spherical harmonic modes

GW Radiation Field

$$h_{ij}^{\text{TT}}(U, \mathbf{X}) = -\frac{4G}{c^2 R} \perp_{ijab}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{abL-2}(U) - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} V_{b)dL-2}(U) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

GW Polarization States

$$h_+ \equiv \frac{1}{2} (P_i P_j - Q_i Q_j) h_{ij}^{\text{TT}}$$
$$h_\times \equiv \frac{1}{2} (P_i Q_j + P_j Q_i) h_{ij}^{\text{TT}}$$

Note: Unit vectors $(\mathbf{P}, \mathbf{Q}, \mathbf{N})$ form an orthogonal triad.

Spherical harmonic modes

GW Radiation Field

$$h_{ij}^{\text{TT}}(U, \mathbf{X}) = -\frac{4G}{c^2 R} \perp_{ijab}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{abL-2}(U) \right. \\ \left. - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} V_{b)dL-2}(U) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

GW modes

$$h_+ - \text{i}h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\theta, \phi)$$

Note: $Y_{-2}^{\ell m}$: Spin-weighted Spherical harmonic basis.

Spherical harmonic modes

GW modes

$$h^{\ell m} = \int d\Omega \left[h_+ - i h_\times \right] \overline{Y}_{-2}^{\ell m}(\theta, \phi)$$

$$h^{\ell m} = -\frac{G}{\sqrt{2} R c^{\ell+2}} \left[U^{\ell m} - \frac{i}{c} V^{\ell m} \right]$$

Note: $(U^{\ell m}, V^{\ell m})$ naturally relate to (U_L, V_L) .

Hereditary Contributions

$$\begin{aligned}
 U_{ij}^{1.5\text{PN}} &= \frac{2GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)}(U - \tau) \left[\ln \left(\frac{c\tau}{2b_0} \right) + \frac{11}{12} \right] \\
 U_{ij}^{2.5\text{PN}} &= \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau \left[M_{a\langle i}^{(3)} M_{j\rangle a}^{(3)} \right] (U - \tau) \right. \\
 &\quad \left. + \frac{1}{7} M_{a\langle i}^{(5)} M_{j\rangle a} - \frac{5}{7} M_{a\langle i}^{(4)} M_{j\rangle a}^{(1)} - \frac{2}{7} M_{a\langle i}^{(3)} M_{j\rangle a}^{(2)} + \frac{1}{3} \epsilon_{ab\langle i} M_{j\rangle a}^{(4)} S_b \right\} \\
 U_{ij}^{3\text{PN}} &= \frac{2G^2 M^2}{c^6} \int_0^{+\infty} d\tau M_{ij}^{(5)}(U - \tau) \left[\ln^2 \left(\frac{c\tau}{2b_0} \right) \right. \\
 &\quad \left. + \frac{11}{6} \ln \left(\frac{c\tau}{2b_0} \right) - \frac{107}{105} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right].
 \end{aligned} \tag{24}$$

Hereditary Contributions

- In the **Multipolar-post-Minkowskian (MPM)** formalism, radiative moments (U_L, V_L) are connected to canonical moments (M_L, S_L). For instance, U_{ij} is given by

$$\begin{aligned}
 U_{ij}(U) = & \overset{\text{Memory term}}{M_{ij}^{(2)}(U)} + \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2b}\right) \right] \overset{\text{Tail term}}{M_{ij}^{(4)}(U - \tau)} \\
 & + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^\infty d\tau M_{a\langle i}^{(3)}(U - \tau) M_{j\rangle a}^{(3)}(U - \tau) + \frac{1}{7} M_{a\langle i}^{(5)} M_{j\rangle a} \right. \\
 & \overset{\text{Instantaneous term}}{\left. - \frac{5}{7} M_{a\langle i}^{(4)} M_{j\rangle a}^{(1)} - \frac{2}{7} M_{a\langle i}^{(3)} M_{j\rangle a}^{(2)} + \frac{1}{3} \epsilon_{ab\langle i} M_{j\rangle a}^{(4)} S_b \right\} \\
 & + \overset{\text{Tail-of-Tail term}}{\frac{2G^2 M^2}{c^6} [\dots]} + \mathcal{O}\left(\frac{1}{c^7}\right)
 \end{aligned}$$

Outline - Lecture 3

- ◆ The **PN-MPM** approach — application to circular binaries
- ◆ Source multipole moments
- ◆ Equations of motion
- ◆ Phasing of Gravitational waves

The **PN-MPM** approach
— application to circular
binaries

The Quadrupole formula

GW Radiation field

$$h_{ij}^{\text{TT}} = -\frac{2G}{c^4 R} \perp_{ijab}(\mathbf{N}) \left\{ \frac{d^2 Q_{ab}}{dT^2} (T - R/c) + \mathcal{O}\left(\frac{1}{c}\right) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

GW energy flux

$$\mathcal{F} \equiv \left(\frac{dE}{dT} \right)^{\text{GW}} = \frac{G}{c^5} \left\{ \frac{1}{5} \frac{d^3 Q_{ab}}{dT^3} \frac{d^3 Q_{ab}}{dT^3} + \mathcal{O}\left(\frac{1}{c^2}\right) \right\}$$

Quadrupole moment tensor

$$Q_{ij}(t) = \int_{\text{source}} d^3 \mathbf{x} \, \rho(\mathbf{x}, t) \left(x_i x_j - \frac{1}{3} \delta_{ij} \mathbf{x}^2 \right)$$

The quadrupole moment

Quadrupole moment tensor

$$Q_{ij}(t) = \int_{\text{source}} d^3\mathbf{x} \rho(\mathbf{x}, t) \left(x_i x_j - \frac{1}{3} \delta_{ij} \mathbf{x}^2 \right)$$

System of N point particles

$$\begin{aligned} Q_{ij}(t) &= \sum_{A=1}^N m_A \int d^3\mathbf{x} \delta(\mathbf{x} - \mathbf{y}_A) x_{\langle i} x_{j \rangle} \\ &= \sum_{A=1}^N m_A y_A^{\langle i} y_A^{j \rangle} \end{aligned}$$

Note: Angular brackets in $x_{\langle i} x_{j \rangle}$ indicate that the quantity is symmetric and trace free in its two indices.

The quadrupole moment

Compact binary

$$Q_{ij} = m_1 y_1^{\langle ij \rangle} + m_2 y_2^{\langle ij \rangle} \quad (\text{Body frame})$$

$$= m \nu x^{\langle ij \rangle} \quad (\text{CoM frame; Exercise})$$

Note:

$$x_{\langle i} x_{j \rangle} = x_{\langle ij \rangle}, m = m_1 + m_2, \nu = \mu m, y_1^i - y_2^i = x^i = r n^i, v^i = \frac{dx^i}{dt} = r \omega \lambda^i$$

$$Q_{ij}^{(2)} = 2\nu m \left(v^{\langle ij \rangle} - \frac{Gm}{r^3} x^{\langle ij \rangle} \right) \quad \text{P1, Tut1}$$

$$Q_{ij}^{(3)} = -8\nu m \frac{Gm}{r^3} v^{\langle i} x^{j \rangle} \quad \text{P1, Tut1}$$

$$(Q_{ij}^{(3)})^2 = 32\nu^2 m^2 \frac{G^3 m^3}{r^5} v^{\langle i} x^{j \rangle} \quad \text{P2, Tut1}$$

Note: $x^i = r n^i$ & $a^i = -\omega^2 x^i = \frac{Gm}{r^3} x^i$ & $v^2 = r^2 \omega^2$

The Quadrupole formula

GW Radiation field

$$h_{ij}^{TT} = -\frac{4G\nu m}{c^4 R} \perp_{ijab} \left(v^{\langle ij \rangle} - \frac{Gm}{r^3} x^{\langle ij \rangle} \right)$$

GW energy flux

$$\mathcal{F} = \frac{32\nu^2}{5} \frac{G^4 m^5}{c^5 r^5} \left[1 + \mathcal{O} \left(\frac{1}{c^2} \right) \right]$$

GW Radiation Field in terms of Source moments

$$h_{km}^{\text{TT}} = (h_{km}^{\text{TT}})_{\text{inst}} + (h_{km}^{\text{TT}})_{\text{tail}}$$

$$(h_{km}^{\text{TT}})_{\text{inst}} = \frac{2G}{c^4 R} \mathcal{P}_{ijklm} \left\{ I_{ij}^{(2)} + \frac{1}{c} \left[\frac{1}{3} N_a I_{ija}^{(3)} + \frac{4}{3} \varepsilon_{ab(i} J_{j)a}^{(2)} N_b \right] + \frac{1}{c^2} \left[\frac{1}{12} N_{ab} I_{ijab}^{(4)} + \frac{1}{2} \varepsilon_{ab(i} J_{j)ac}^{(3)} N_{bc} \right] \right. \\ \left. + \frac{1}{c^3} \left[\frac{1}{60} N_{abc} I_{ijabc}^{(5)} + \frac{2}{15} \varepsilon_{ab(i} J_{j)acd}^{(4)} N_{bcd} \right] + \frac{1}{c^4} \left[\frac{1}{360} N_{abcd} I_{ijabcd}^{(6)} + \frac{1}{36} \varepsilon_{ab(i} J_{j)acde}^{(5)} N_{bcde} \right] \right\}$$

$$(h_{km}^{\text{TT}})_{\text{tail}} = \frac{2G}{c^4 R} \frac{2Gm}{c^3} \mathcal{P}_{ijklm} \int_0^{+\infty} d\tau \left\{ \ln \left(\frac{\tau}{2b_1} \right) I_{ij}^{(4)}(T_R - \tau) + \frac{1}{3c} \ln \left(\frac{\tau}{2b_2} \right) N_a I_{ija}^{(5)}(T_R - \tau) \right. \\ \left. + \frac{4}{3c} \ln \left(\frac{\tau}{2b_3} \right) \varepsilon_{ab(i} N_b J_{j)a}^{(4)}(T_R - \tau) \right\} .$$

$$b_1 \equiv b e^{-11/12} , \quad b_2 \equiv b e^{-97/60} , \quad b_3 \equiv b e^{-7/6}$$

$$U = t - \frac{r}{c} - \frac{2GM}{c^3} \ln \left(\frac{r}{cb} \right) + \mathcal{O}(\varepsilon^5)$$

Radiation Reaction

Newtonian Orbital Energy

$$E = -\frac{1}{2}\nu\frac{Gm^2}{r}$$

Invoking energy Balance : $\frac{dE}{dt} = -\mathcal{F}$

$$\dot{r} = \left(\frac{dE}{dt} / \frac{dE}{dr} \right) = -\frac{64}{5}\nu\frac{G^3m^3}{r^3c^5}$$

$$\dot{\omega} = \left(\frac{d\omega}{dr} / \frac{dr}{dt} \right) = \frac{96}{5}\nu\left(\frac{G^7m^7}{c^{10}r^{11}}\right)^{1/2}$$

GW Phasing Formula

**[PN expansion
parameter :
 $x=v^2$]**

$$\begin{aligned}
 \psi = \psi_0 - \frac{x^{-5/2}}{32\nu} & \left\{ 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu \right) x - 10\pi x^{3/2} \right. \\
 & + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2 \right) x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu \right) \pi x^{5/2} \ln x \\
 & + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_E - \frac{856}{21} \ln(16x) \right. \\
 & \quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2 \right) \nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3 \right] x^3 \\
 & + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2 \right) \pi x^{7/2} \\
 & + \left[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_E - \frac{45245}{756}\pi^2 - \frac{252755}{2646} \ln 2 - \frac{78975}{1568} \ln 3 - \frac{9203}{252} \ln x \right. \\
 & \quad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_E + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323} \ln 2 + \frac{78975}{392} \ln 3 - \frac{244493}{1323} \ln x \right) \nu \\
 & \quad \left. + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2 \right) \nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4 \right] x^4 \\
 & + \left[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_E + \frac{80}{3}\pi^2 + \frac{856}{21} \ln(16x) \right. \\
 & \quad \left. + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2 \right) \nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3 \right] \pi x^{9/2} + \mathcal{O}(x^5) \Bigg\} ,
 \end{aligned}$$

[Order counting terminology : nPN := x^n] 43

[Blanchet Living Review]

post-Newtonian approx.

- ◆ Taylor series expansions in small parameters such as internal source velocities
- ◆ Point particles, multipolar expansions etc.
- ◆ High order computations (to gain accuracy but nontrivial due to divergences)

Detector	LIGO/Virgo		ET		LISA	
Masses ($M_{\odot}$)	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
PN order	cumulative number of cycles					
Newtonian	2 562.599	95.502	744 401.36	37.90	28 095.39	9.534
1PN	143.453	17.879	4 433.85	9.60	618.31	3.386
1.5PN	−94.817	−20.797	−1 005.78	−12.63	−265.70	−5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	−8.105	−4.604	−17.01	−3.42	−12.47	−1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	−0.627	−0.689	−0.93	−0.59	−0.91	−0.383
4PN	−0.107	−0.064	−0.12	−0.04	−0.12	−0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

[Order counting terminology : $n\text{PN} := v^{2n}$]

[Blanchet+, arXiv:2304.11185]

Current Status

GW phasing

- ◆ 4.5PN **non-spinning, quasi-circular** [..., Blanchet+, arXiv:2304.11185]
- ◆ Quasi-circular SO (4PN), SS (3PN), SSS (3.5PN) [..., Mishra+, arXiv:1601.05588]
- ◆ 3PN **non-spinning, eccentric** [..., B Moore+, arXiv:1604.00304]
- ◆ 3PN fluxes **spinning, eccentric** [..., Q Henry+, arXiv:2308.13606]
- ◆ 3PN **spinning, eccentric** [..., **Sridhar+**, arXiv:2412.10909]
- ◆ Tidal effects **non-spinning, circular** (7.5PN) [..., Q Henry, arXiv:2005.13367]
spinning, circular (6.5PN) [..., Castro+, arXiv: 2204.12510]

GW Amplitude

- ◆ 4PN quadrupole mode **non-spinning, quasi-circular** orbits [..., L Blanchet+, arXiv:2304.11186].
- ◆ 3.5PN all modes **spinning, quasi-circular** [..., Q Henry+, arXiv:2210.15602]
- ◆ 3PN all modes **non-spinning, eccentric** [..., **Boetzel+**, arXiv:1904.11814, **Ebersold+**, arXiv:1906.06263]
- ◆ 3PN all modes **spinning, eccentric** [...**Paul+**, arXiv:2211.04155, Q Henry+, arXiv:2308.13606]

IMR prescriptions (current Status)

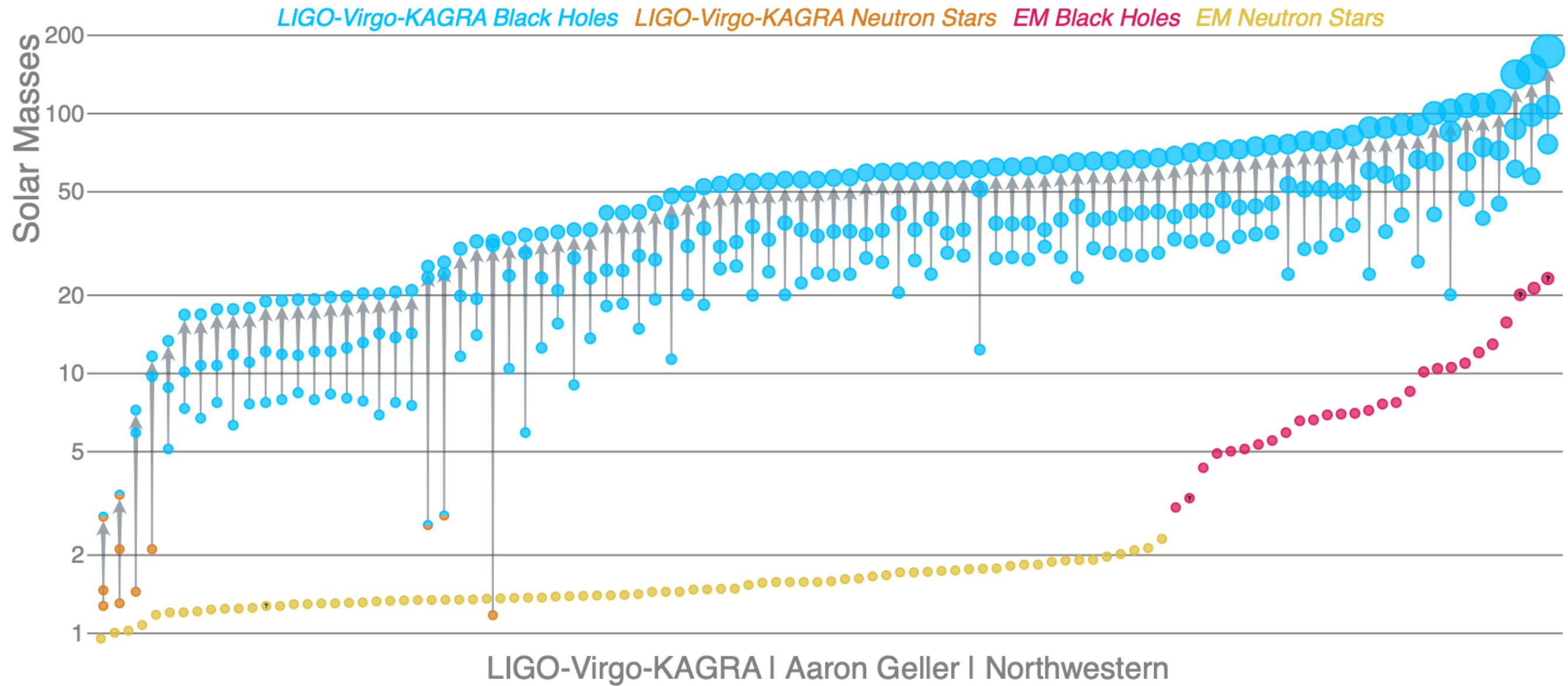
EOB IMR models

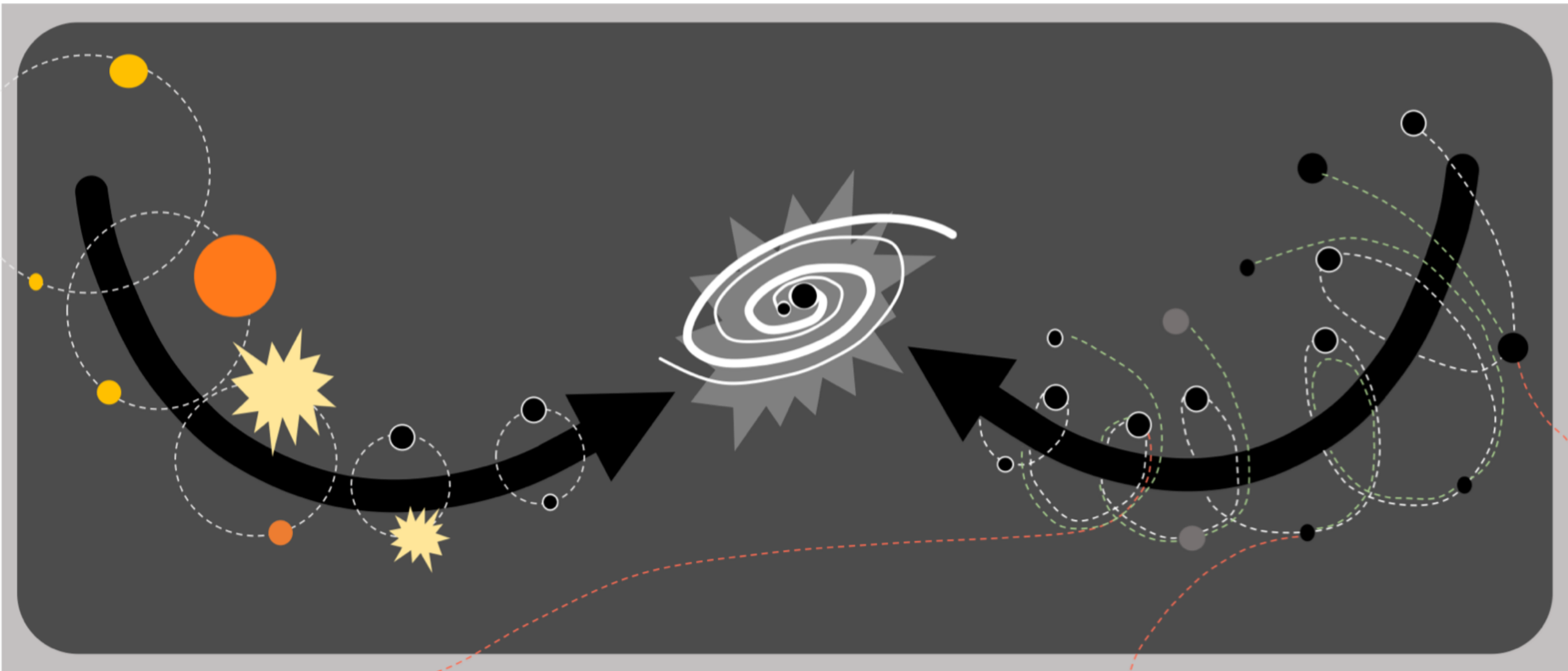
- ◆ SEOBNRv5PHM [..., Ramos-Buades+, arXiv:2303.18046]
- ◆ SEOBNRv4EHM [..., Ramos-Buades+, arXiv:2112.06952]
- ◆ TEOBResumS [..., Nagar+, arXiv:1806.01772]
- ◆ TEOBResumSGeneral [Nagar+, arXiv:2101.08624]

Phenom and other IMR models

- ◆ PhenomPv3HM [..., Khan+, arXiv:1911.06050]
- ◆ PhenomXPHM [..., Pratten+, arXiv:2004.06503]
- ◆ NRSur7dq4 [..., Varma+, arXiv:1905.09300]
- ◆ NRSur2dq1Ecc [..., Islam+, arXiv:2101.11798]
- ◆ EOBNRE_S [..., Yun+, arXiv:2104.03789]
- ◆ ENIGMA [..., **Paul+**, arXiv:2409.13866, **Maurya+** arXiv:2510.00116]
- ◆ Non-spinning EccentricIMR Models [..., **Chattaraj+**, arXiv:2204.02377, **Manna+**, arXiv:2409.10672]

Masses in the Stellar Graveyard





Thank You!

[Image Credits : Isobel Romero-Shaw]

additional slides

Source Multipoles moments

$$I_L(t) = \int d^3x \left[\hat{x}_L \sigma(t, \mathbf{x}) + \frac{\hat{x}_L \mathbf{x}^2}{2(2\ell + 3)c^2} \frac{\partial^2 \sigma(t, \mathbf{x})}{\partial t^2} - \frac{4(2\ell + 1)\hat{x}_{iL}}{(\ell + 1)(2\ell + 3)c^2} \frac{\partial \sigma_i(t, \mathbf{x})}{\partial t} \right] + O(\varepsilon^4)$$

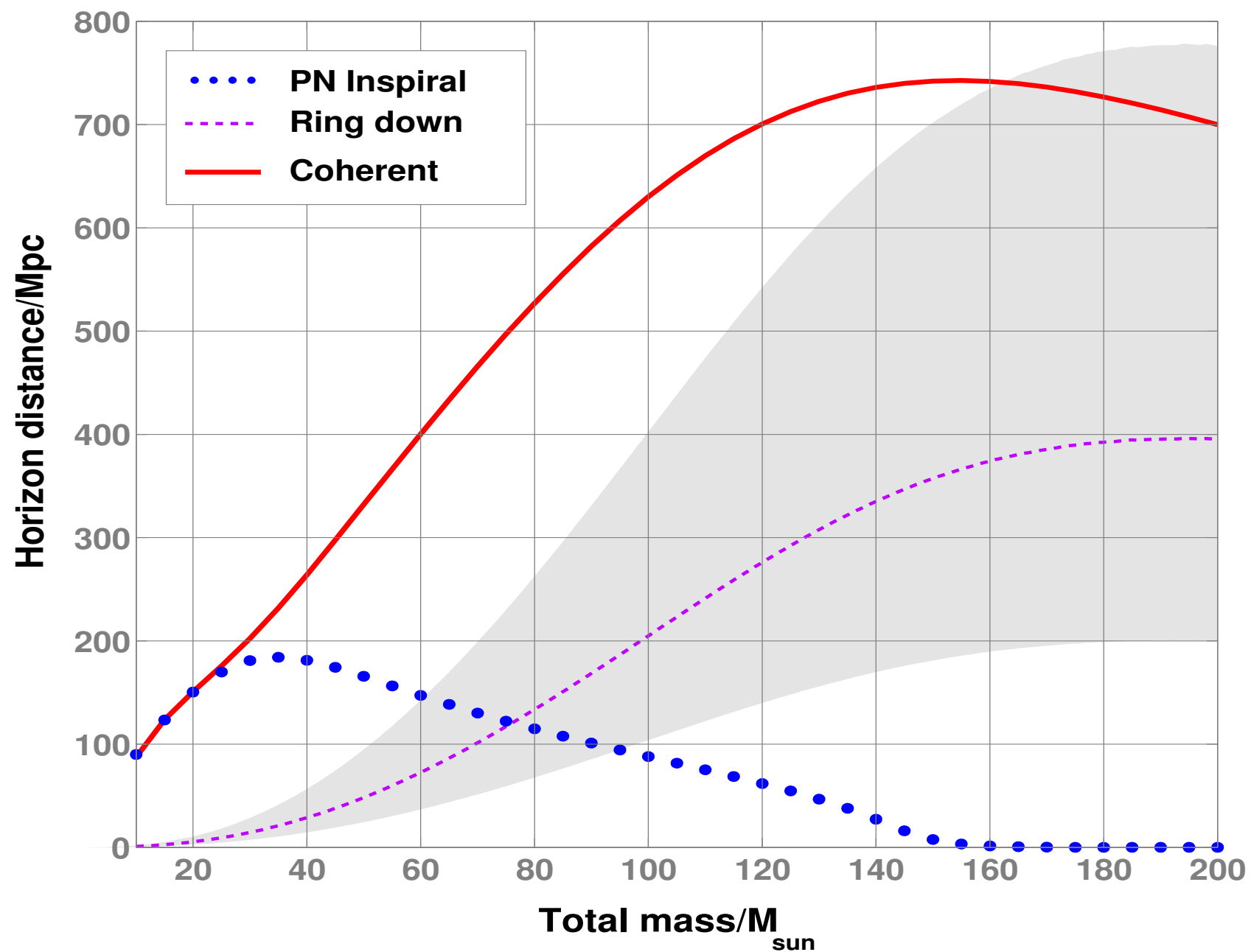
$$\sigma(t, \mathbf{x}) \equiv \frac{T^{00}(t, \mathbf{x}) + T^{ss}(t, \mathbf{x})}{c^2}$$

$$\sigma_i(t, \mathbf{x}) \equiv \frac{T^{0i}(t, \mathbf{x})}{c},$$

$$\sigma_{ii}(t, \mathbf{x}) \equiv T^{ij}(t, \mathbf{x}).$$

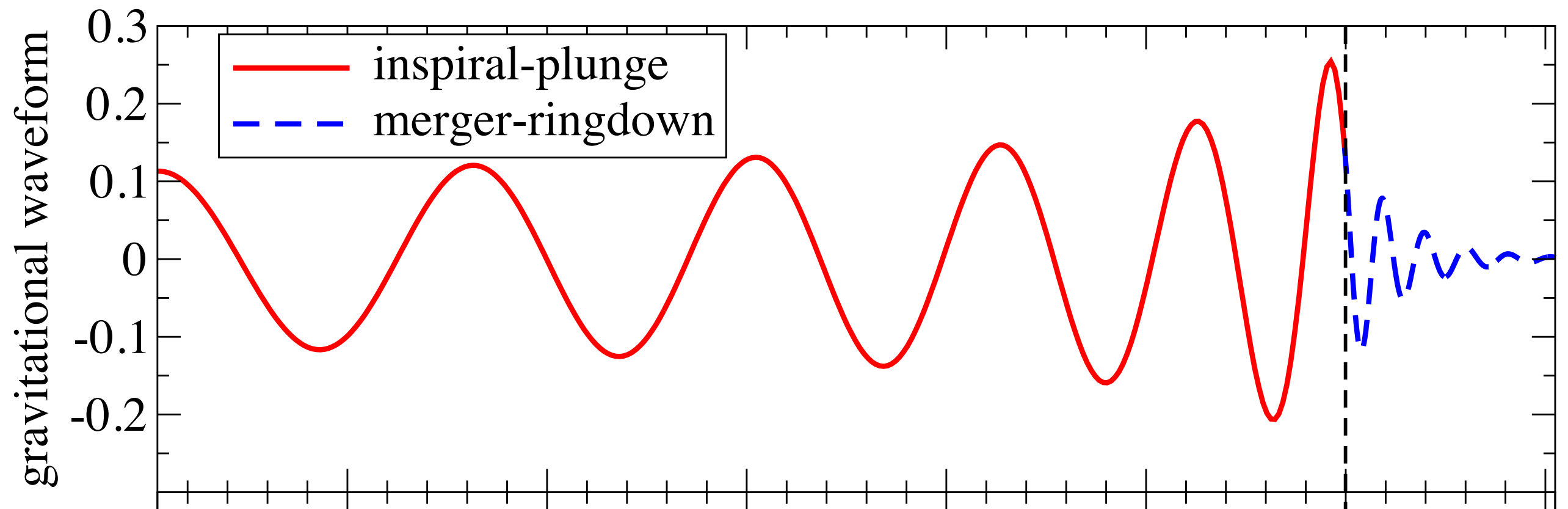
$$T^{\mu\nu}(\mathbf{x}, t) = \sum_{A=1}^N m_A \frac{dy_A^\mu}{dt} \frac{dy_A^\nu}{dt} \frac{1}{\sqrt{-g}} \frac{dt}{d\tau} \delta(\mathbf{x} - \mathbf{y}_A(t))$$

Coherent inputs

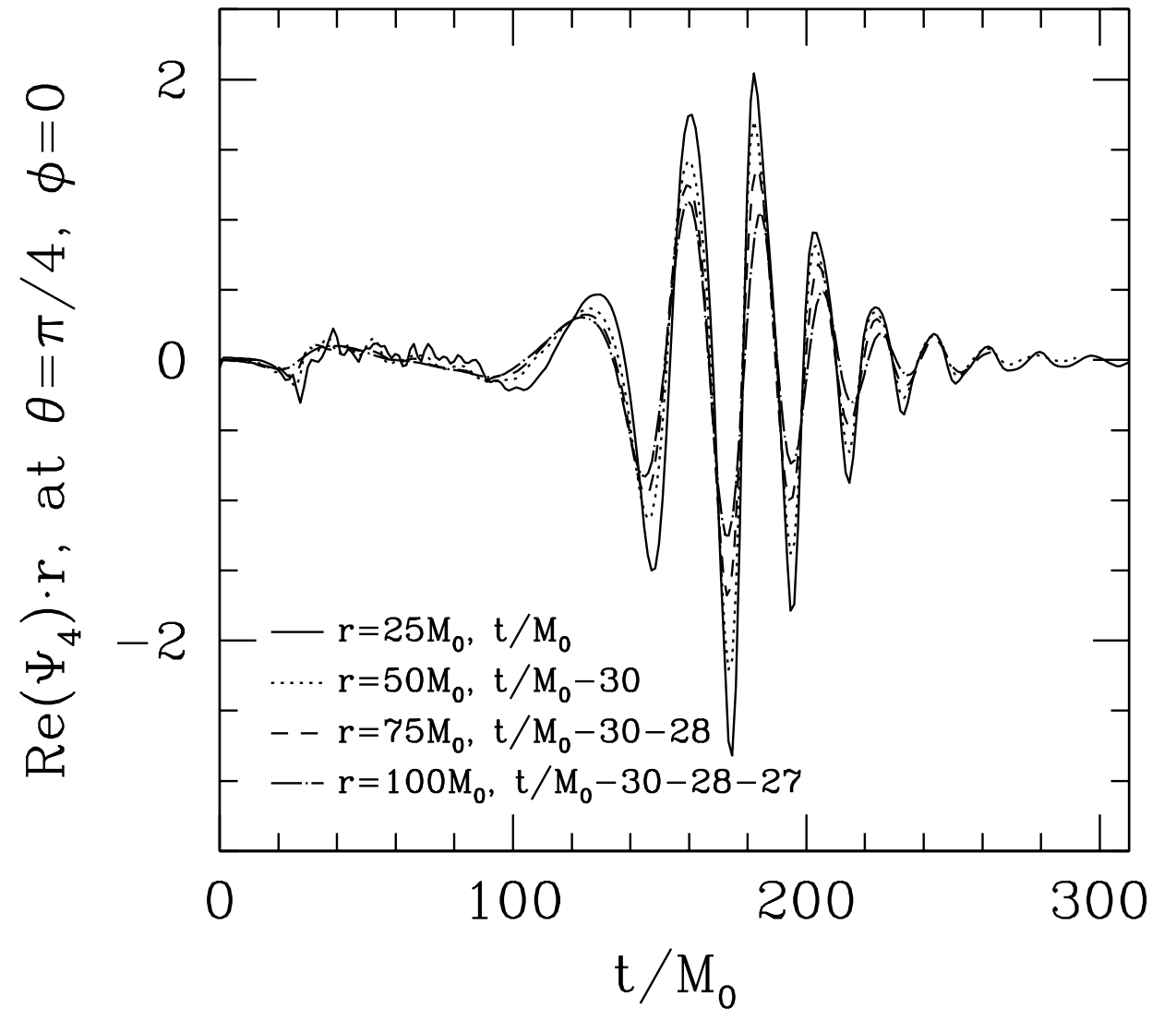
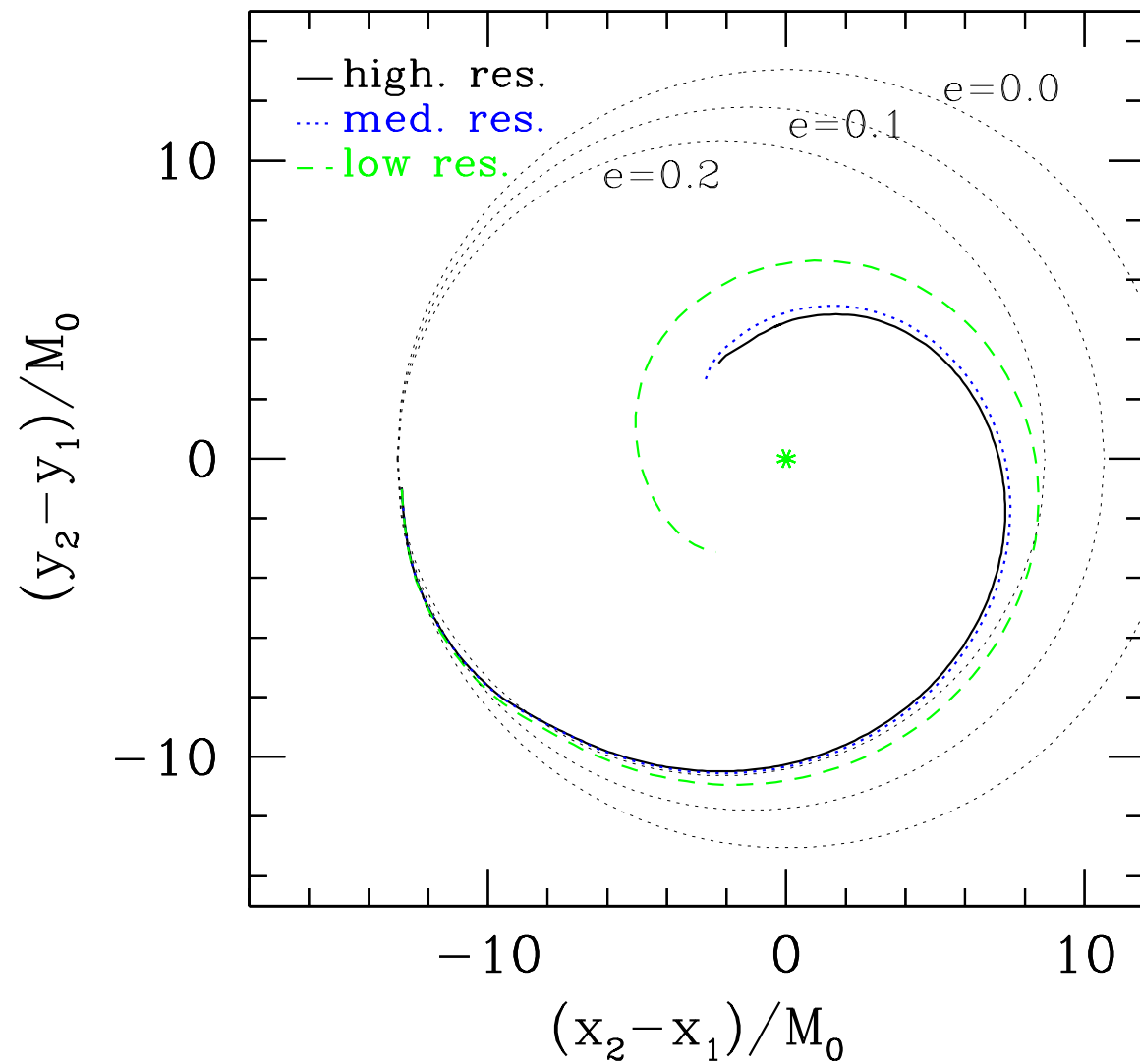


[Ajith ('07)]

The EOB trick

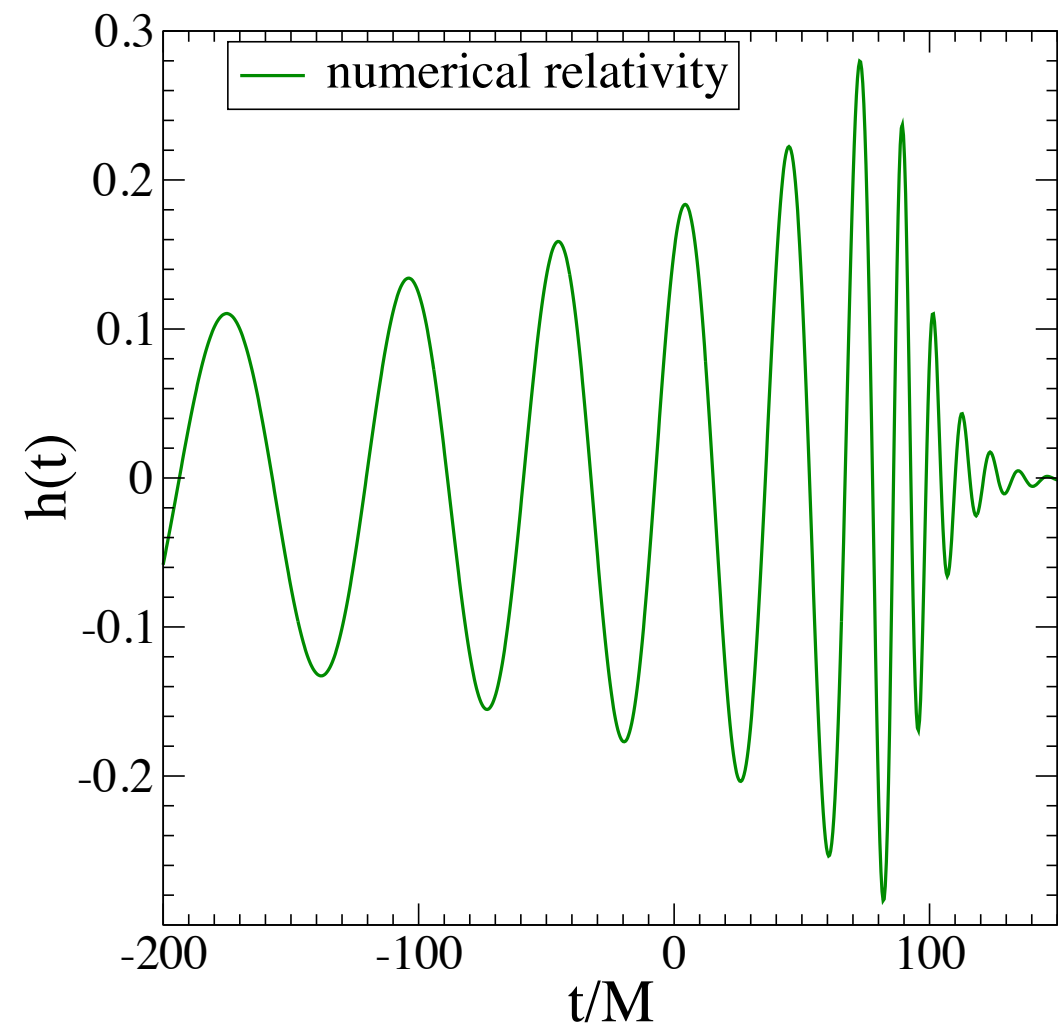
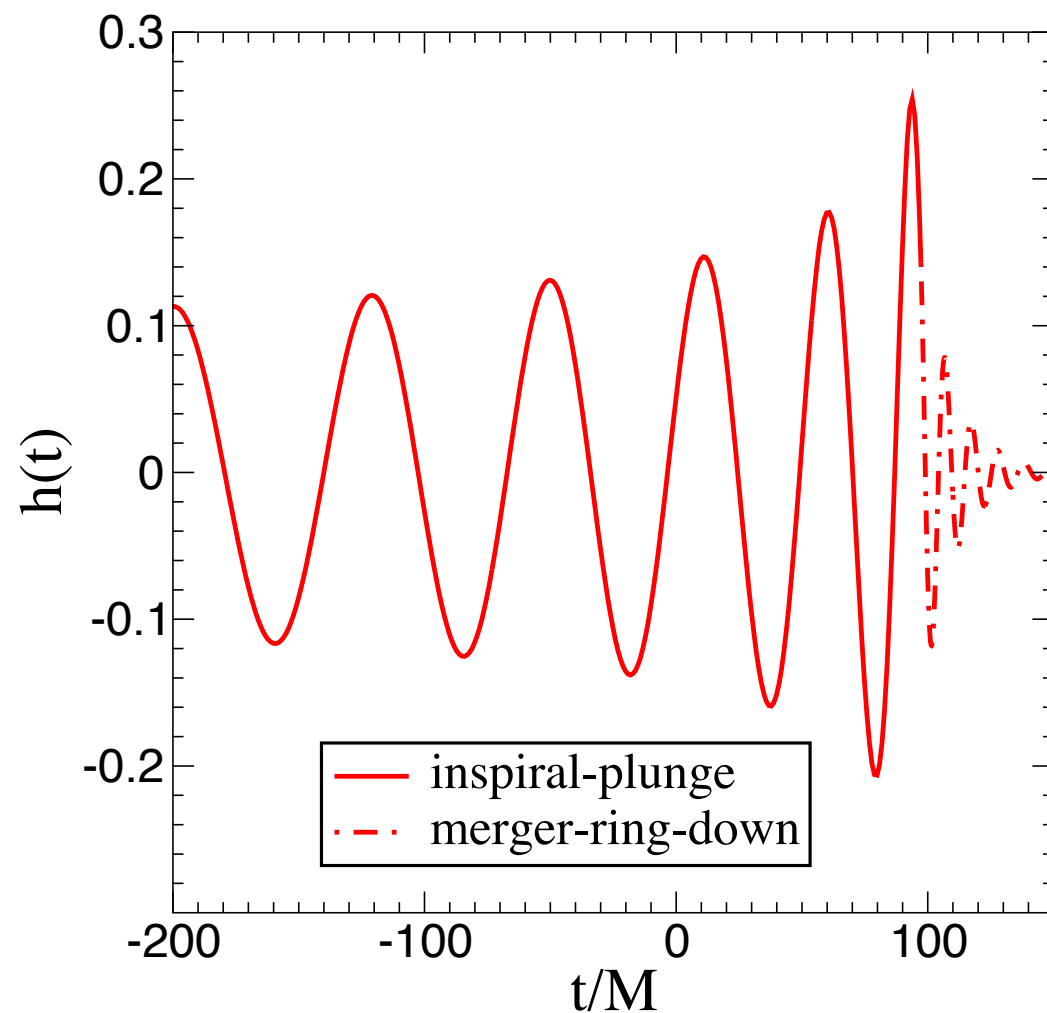


Numerical Relativity



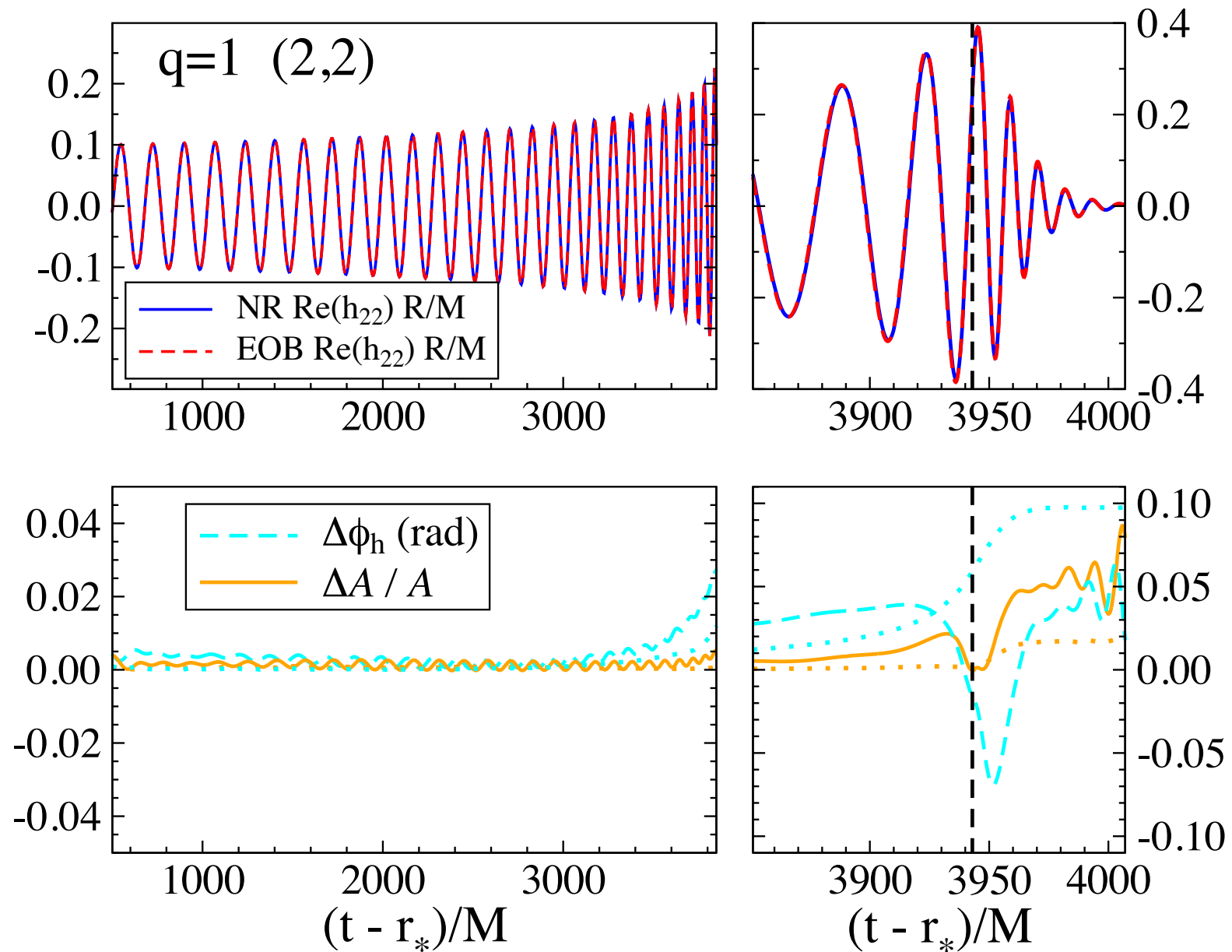
Pretorius (2005)

The EOB trick

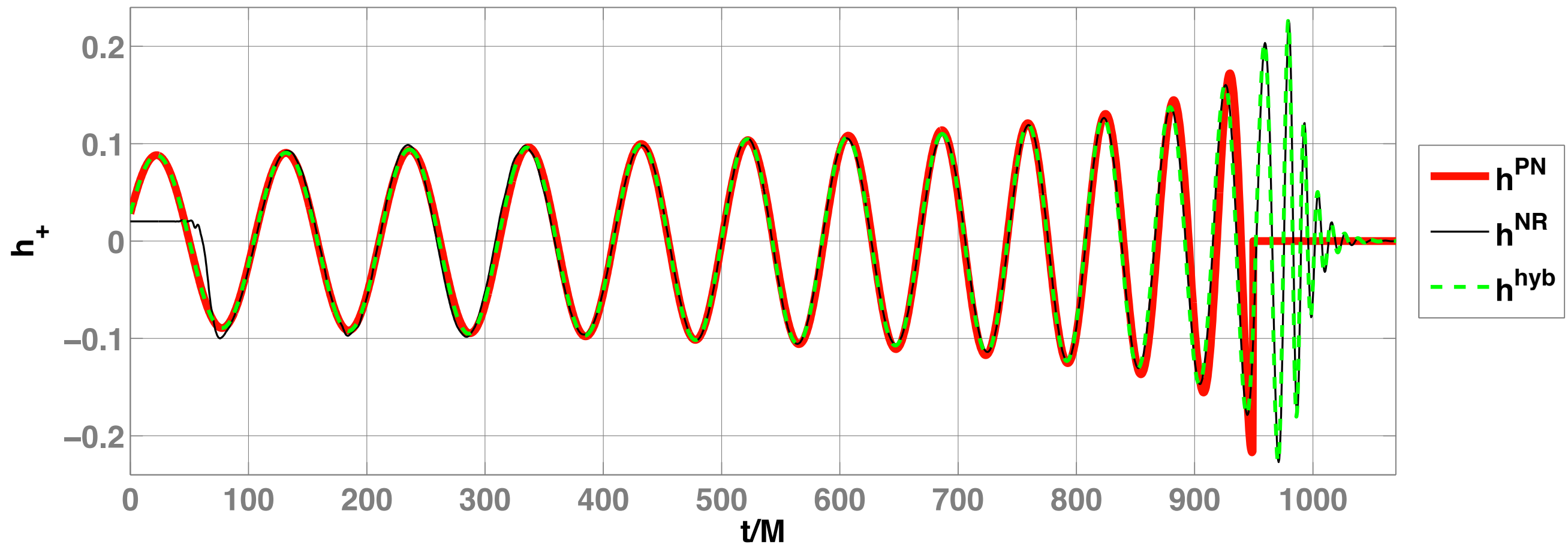


[Buonanno, arXiv:0709.4682 (2007)]

Perfecting the trick



Phenomenological approach

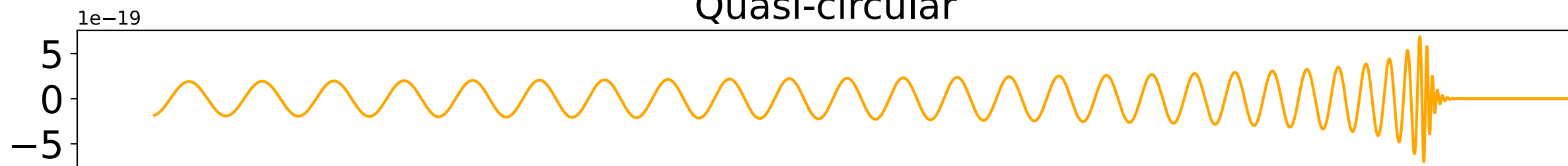


[Ajith et al., CQG 24:S689-S700 (2007)]

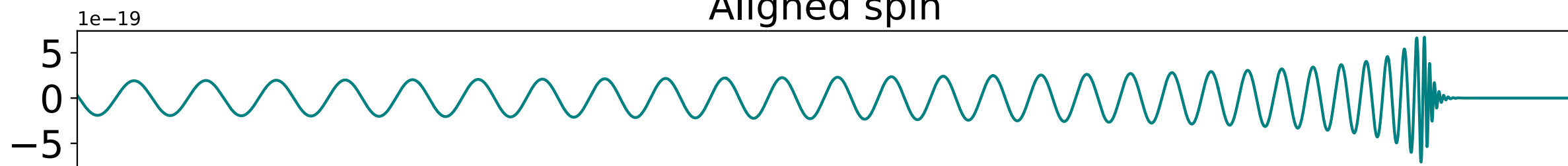
Public Numerical Relativity Simulations

- ◆ **SXS catalog** — over 2000 simulations, $q=1-10$; including **20 eccentric, non-spinning** ($q=1-3$) simulations [..., [Boyet+, arXiv:1904.04831](#)]; used in analysing LVK events
- ◆ **Georgia Tech (now MAYA catalog)** — over 600 simulations including 92 eccentric, spinning simulations including **7 with both eccentricity and spin-precession** [..., [Ferguson+, arXiv:2309.00262](#)]; used in analysing LVK events
- ◆ **RIT catalog** — over **1800** simulations; including close to 800 **eccentric** simulations albeit too short (2-4 cycles); used in analysing GW190521 (first IMBH event) [..., [Healy+, arXiv:2202.00018](#)]
- ◆ **BAM catalog** — 80 single-spin, (quasi-circular) precessing configurations at 4 different mass ratios [..., [Healy+, arXiv:2303.05419](#)]

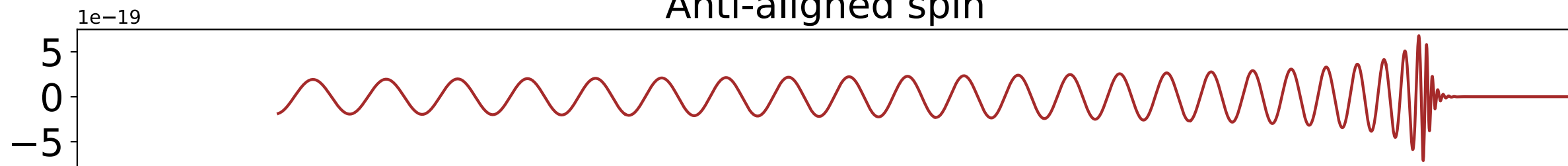
Quasi-circular



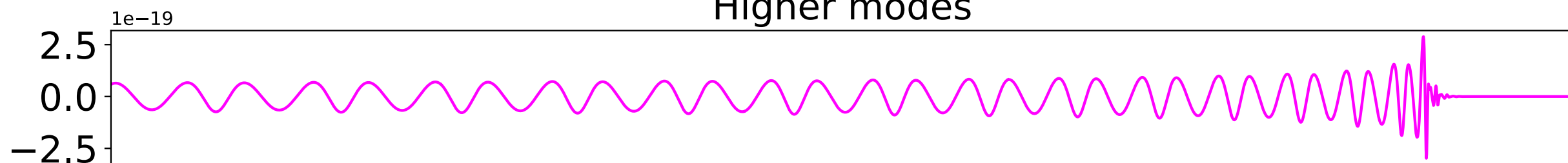
Aligned spin



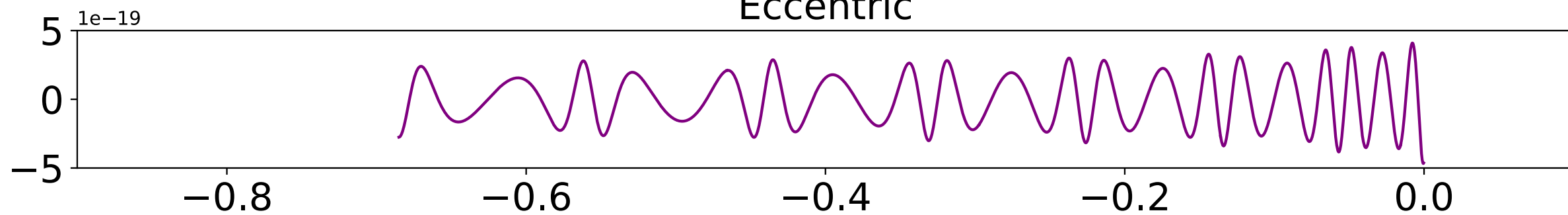
Anti-aligned spin



Higher modes



Eccentric



Time [s]